Computational Methods to Exploring Structures in EvenQuads

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Abstract

EvenQuads is a mathematical card game created by Dr. Lauren Rose in collaboration with the American Mathematical Society (AMS) and the Association for Women in Mathematics (AWM) to make the exploration of mathematical structure, combinatorics and geometry engaging and accessible. In this project we explore the combinatorial and geometric structure of *quads* - sets of four binary vectors in F_2^n whose sum is zero. These structures appear naturally in affine geometry and combinatorics. We explore both the construction of *quad-rich* sets and sequences - those that maximize the number of quads - using a combination of combinatorics, geometry, and computational tools. We examine questions like: What is the probability of getting a quad in a k-card layout in n even dimensions? What kinds of structural patterns yield the most quads of each type? And how do mathematical sequences like the Thue-Morse sequence influence quad formation compared to random sampling?

Quads visualisation using Principal component analysis (PCA) in MATLAB

PCA is a commonly used dimensionality reduction technique that reduces the number of variables into principal components that represent the directions of maximum variance. Our goal is to visualise for a given deck size, the maximum number of Quads of each type and to explore structural patterns in Quads by clustering them based on similarity in their A/H/D patterns. To do this, we first identified all Quads within a given sample of card layouts and encoded each Quad according to its type (e.g., AAD, HHA, etc.). Using PCA, we projected the six-dimensional data into a 2D space where every card appears as a point. A Quad is represented by a 4-sided polygon. The colors indicate the cluster each Quad was assigned to. For a polygon with smaller sides, it means the cards associated with that Quad is likely to have more attributes that are in the same state. This visualization helped reveal potential structure. We tried to see if some types of Quads may form more frequently or naturally than others.

To explore the structure of *packed sets*, subsets of cards that contain the maximum number of quads, we performed an empirical simulation across deck sizes k . For each value of k we generated 500 random subsets and computed the number of valid quads in each. We then retained the sample with the highest number of quads and plotted this in 2D space. This visualization helps us approximate the growth of the maximum number of quads as the function F(k).

The Game and Definitions

Quad: A quad is a set of four cards that satisfy the following condition: for each attribute (color, shape, number) the values are either all the same, all different, or half-and-half (two of one kind and two of another).

Algebra: Algebraically, cards are represented as binary vectors a, b, c, d in $\mathbb{Z}_2^n$. We can assign each state a binary number as follows:

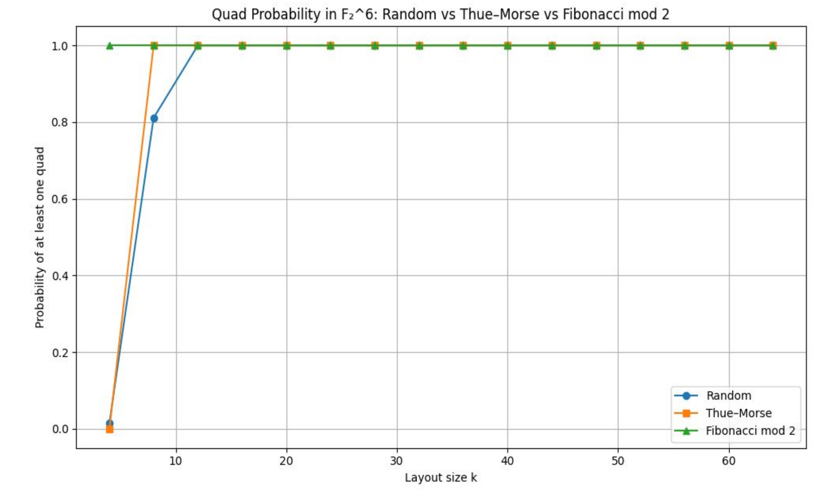
Numbers: 1 = 00, 2 = 01, 3 = 10, 4 = 11
Colors: red = 00, green = 01, blue = 10, yellow = 11
Shapes: circle = 00, triangle = 01, square = 10, heart = 11

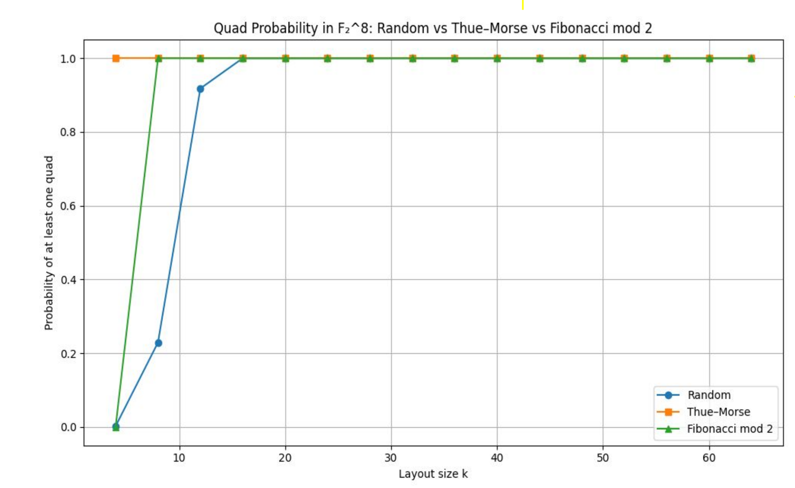
Each EvenQuads card corresponds to a point in AG(n,2).

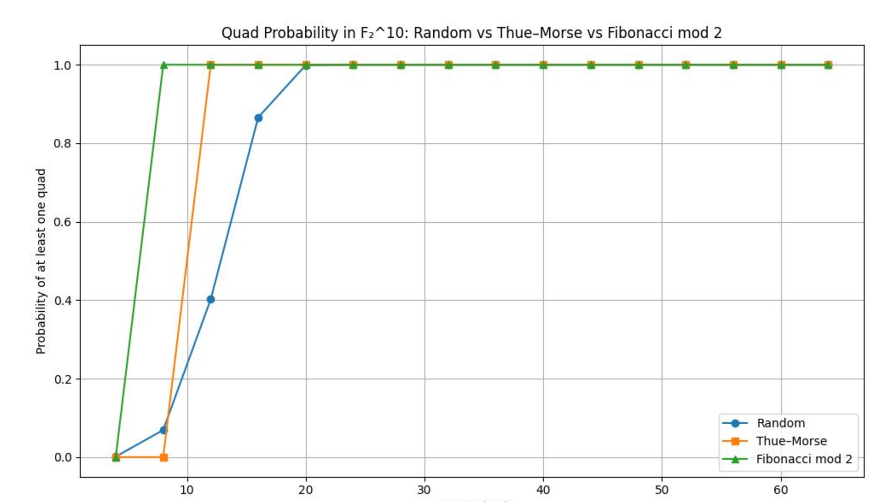
Quad Types (by Attributes): Quads are classified by attribute patterns—e.g., ADD (one attribute all alike, others all different), HHH (half-half on each attribute).

Packed Set: A packed set is a subset of cards of size k that contains the maximum possible number of quads, denoted by F(k).

Findings







- For all n values in small k's, random layouts consistently underperform against Thue–Morse and Fibonacci mod 2 layouts.
- The Thue–Morse probability rises to 1 after its specific threshold, which shifts with n. For example, it reaches 1 around k=8 for n=6 and around k=12 for n=10.These shifts underscore the importance of both sequence structure and vector space size in determining the likelihood of quad formation, demonstrating how deterministic properties can accelerate quad formation compared to random sampling.

Methods

We evaluated the likelihood of quad formation in layouts of size $k \in \{4, 5, \ldots, 64\}$ drawn from binary vector spaces F_2^n for even values of n using NumPy arrays or bit-strings.

Quad Detection
To detect quads, we used a brute-force approach checking all 4-element subsets $\{v_1, v_2, v_3, v_4\} \subseteq S$ for the condition: $v_1 + v_2 + v_3 + v_4 \equiv 0 \pmod 2$.
To handle the combinatorial growth, we applied performance optimizations using Numba and multiprocessing, allowing us to process thousands of layouts across varying k and n.

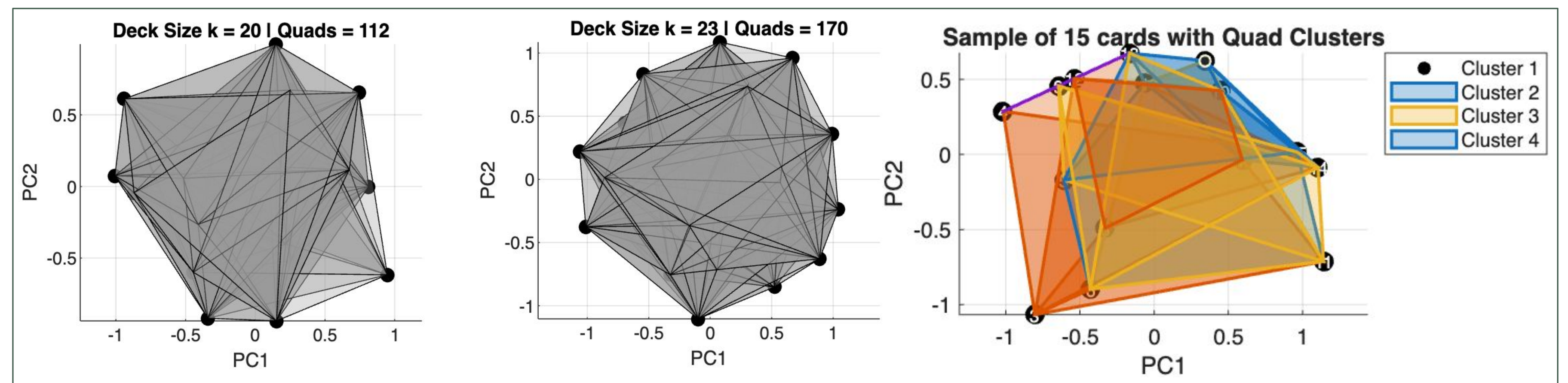
Probability estimation via **Monte Carlo sampling**

For each k, we generated 1000–5000 random k-element subsets (without replacement) from the full set of 2^n vectors, recording the presence or absence of a quad in each.

Structured Layout Evaluation
We also tested structured layouts based on known sequences:

- Thue–Morse: for its self-similar, anti-repetitive properties
- Fibonacci mod 2: for contrast with a simple linear recurrence

These were padded or truncated to match the same k, and analyzed using the same detection method.



While we did not attempt to formally prove the values of F(k), our simulations align closely with theoretical values for small values of k as derived in *Maximum Number of Quads* (Byrapuram et al., 2024) which establishes exact values for F(k) in certain cases. Although our observed maximums did not always match the theoretical maximums, this is likely due to computational limitations. We only performed 500 trials per k, and the MATLAB code used to compute quads became increasingly time-consuming for a larger number of trials. These constraints may explain the slight discrepancies in values. Nonetheless, our results demonstrate that sampling-based approaches are promising tools for approximating or discovering packed sets and understanding quad-rich regions in the combinatorial landscape of EvenQuads.

Acknowledgements

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